

EFFECTIVE CHIRAL MODEL OF A PLANE-SYMMETRIC GRAVITATIONAL FIELD: PROPERTIES AND EXACT SOLUTIONS

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An effective chiral model of a plane-symmetric gravitational field is considered. Isometries of the target space of the model are described and exact solutions corresponding to the isometric ansatz method are obtained. New exact solutions are found using the method of functional parameters. The solutions obtained are Bäcklund transforms of solutions of the d'Alembert equation to those of the Einstein equations.

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1 Introduction

One of the most interesting nonlinear field models is the chiral (bosonic) nonlinear sigma model which has a lot of possibilities to be investigated by geometrical methods [1]. It was pointed out that a certain class of Einstein gravitational field equations can be represented as the dynamical equations of a nonlinear sigma model (NSM) with a special choice of a target space [2, 3, 4]:

$$\frac{1}{\sqrt{|g|}} \partial_i (\sqrt{|g|} \varphi_A^i) - \Gamma_{C,AB} \varphi_i^B \varphi_{,k}^C g^{ik} = 0, \quad (1)$$

which follow from the action

$$\mathcal{S} = \int_{\mathcal{M}} \sqrt{|g|} d^4 x \left\{ \frac{1}{2} h_{AB}(\varphi) \varphi_{,i}^A \varphi_{,k}^B g^{ik} \right\}. \quad (2)$$

The notations corresponded to those in [5]. Using this representation, one can apply the isometric ansatz method [6] with the aim to find exact solutions describing gravitational fields. On the other hand, a direct analysis of the transformed Einstein equations is possible as well.

In the present article isometries of the target space of the effective chiral model of a plane-symmetric gravitational field are considered and exact solutions corresponding to the isometric ansatz method are obtained. New exact solutions for the model under consideration are found by the method of functional parameters. The solutions obtained are the results of Bäcklund transformations of the D'Alembert equation to the Einstein equations.

2 Chiral effective model

The effective chiral model of a plane-symmetric gravitational field

$$ds^2 = A(dx^1)^2 + 2Bdx^1 dx^2 + C(dx^2)^2$$

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$$-D[(dx^3)^2 - (dx^4)^2] \quad (3)$$

is considered. The metric coefficients A, B, C, D depend only on x^3, x^4 .

The vacuum Einstein equation

$$R_{ik} = 0 \quad (4)$$

for the space-time (3), as was shown in [3], may be written as a four-component nonlinear sigma model for the fields $\varphi^1 = \psi$, $\varphi^2 = \theta$, $\varphi^3 = \chi$, $\varphi^4 = \phi$, defined on the two-dimensional space-time

$$dS^2 = (dx^3)^2 - (dx^4)^2. \quad (5)$$

The metric of the target space for the effective chiral model can be reduced to the form [3]

$$h_{IK} = e^\psi \begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \sinh^2 \theta & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \quad (6)$$

The metric coefficients should be connected with the chiral field by

$$\begin{aligned} A &= -e^\psi (\cos \chi \sinh \theta + \cosh \theta), \\ B &= e^\psi \sin \chi \sinh \theta, \\ C &= e^\psi (\cos \chi \sinh \theta - \cosh \theta), \\ D &= e^\phi. \end{aligned} \quad (7)$$

The vacuum Einstein equations (4) may be written as the dynamic equations [3]

$$\begin{aligned} \Delta e^\psi &= 0, \\ \Delta \theta + (\psi_3 \theta_3 - \psi_4 \theta_4) - \frac{1}{2} (\chi_3^2 - \chi_4^2) \sinh 2\theta &= 0, \\ \Delta \chi + 2(\theta_3 \chi_3 - \theta_4 \chi_4) \coth \theta + (\psi_3 \chi_3 - \psi_4 \chi_4) &= 0, \\ \Delta \phi + \frac{1}{2} (\psi_3^2 - \psi_4^2) - \frac{1}{2} (\chi_3^2 - \chi_4^2) \sinh^2 \theta - \frac{1}{2} (\theta_3^2 - \theta_4^2) &= 0, \end{aligned} \quad (8)$$

$$\Delta \equiv \partial_{33} - \partial_{44}. \quad (9)$$

To finish our construction, it is necessary to check the reversibility of the transformation (7). The determinant of the transformation Jacobian is equal to

$$\det J = \sinh \theta. \quad (10)$$

So one should assume the only restriction following from the condition (10) to transfer the analysis from (4) to (8).

There are a lot of verified methods of studying non-linear sigma models. In this work we try to apply to the effective chiral model analysis the method of an isometric ansatz based on its symmetry properties.

3 Isometries of the effective chiral model

The allowed isometric motions in the chiral space and their connection with the space-time symmetry will be examined.

The isometries of the target space (6) can be obtained by solving the Killing equations

$$\zeta_{A;B} + \zeta_{B;A} = 0. \quad (11)$$

(11) results in the linearly independent Killing vectors

$$\zeta_1 = (0, \cos \chi, -\sin \chi \coth \theta, 0), \quad (12)$$

$$\zeta_2 = (0, \sin \chi, \cos \chi \coth \theta, 0), \quad (13)$$

$$\zeta_3 = (0, 0, 1, 0), \quad \zeta_4 = (0, 0, 0, 1). \quad (14)$$

The Killing vectors of the space-time (5) are

$$\xi_1^i = \delta_4^i, \quad \xi_2^i = \delta_3^i, \quad \xi_3^i = x^3 \delta_4^i - x^4 \delta_3^i.$$

To study the symmetry properties of the model, we use the isometric ansatz introduced by G.G. Ivanov [6]

$$\xi_a^i \varphi_i^A = K_a^\alpha \zeta_\alpha^A. \quad (15)$$

Here φ_i^A mean the derivatives of the field φ^A in x^i and the constants K_a^α are the expansion coefficients.

Consider a subgroup of the basic space-time isometry associated with $\tilde{\xi}_1$. The ansatz may be written as

$$\begin{aligned} \psi_4 &= 0, & \phi_4 &= d, & \theta_4 &= -\tilde{a} \sin \tilde{\chi}, \\ \tilde{\chi}_4 &= -\tilde{a} \coth \theta \cos \tilde{\chi} + m; \end{aligned} \quad (16)$$

$$\begin{aligned} \tilde{a} &= \sqrt{a^2 + b^2}, & \tilde{\chi} &= \chi + \alpha, & a &= K_1^1, \\ b &= K_1^2, & \alpha &= \arctan(b/a), & m &= K_1^3, \\ d &= K_1^4. \end{aligned} \quad (17)$$

Solutions will be sought in the case $m = 0$, $d \neq 0$, $\tilde{a} \neq 0$.

A solution to the ansatz may be written in the form

$$\begin{aligned} \psi &= \psi(x^3), & \phi &= d \cdot x^4 + f(x^3), \\ \theta &= \text{Arcosh} \left\{ \frac{\sqrt{\tilde{a} X^2}}{\tilde{a} X} \cosh \left\{ \tilde{a} [x^4 - B(x^3)] \right\} \right\}, \\ \chi &= \arccos \left(\frac{1}{\tilde{a} X \sinh \theta} \right) - \alpha, \\ X &= X(x^3). \end{aligned} \quad (18)$$

4 Exact solution

The ansatz solution singles out some class of chiral fields for which it is possible to solve the field equations (8). To this end it is necessary to determine the parametric functions $X = X(x^3)$, $B = B(x^3)$ and the fields $\psi(x^3)$, $\phi(x^3, x^4)$. Substitution of the ansatz solution to the effective chiral model equation leads to separation of variables connected with a dependence of the functions X and B on only x^3 . A necessary condition for the existence of solutions to the effective chiral model in the form (18) is $X(x^3) = \alpha$ and $B(x^3) = \beta x^3$ ($\alpha, \beta = \text{const}$). Further research has shown that the sufficient condition makes narrower the class of possible effective chiral model solution, which is rather narrow itself. So the only case when the ansatz solution (18) satisfies the effective chiral model equation (8) is as follows:

$$\begin{aligned} \psi &= c, & \phi &= d \cdot x^4 + p x^3, \\ \theta &= \text{Arcosh} \left\{ 1/\sqrt{\tilde{a}} \cosh \left(\tilde{a} [x^4 - x^3] \right) \right\}, \\ \chi &= \arccos [1/(\tilde{a} \sinh \theta)] - \alpha, \end{aligned} \quad (19)$$

c and p being constants.

The metric coefficients found according to (7) are

$$\begin{aligned} A &= -\frac{1}{\tilde{a}} e^{-c} \left[\frac{a}{\tilde{a}} + \sqrt{\tilde{a}^2 + 1} \right. \\ &\quad \times \left. \left\{ \frac{b}{\tilde{a}} |\sinh[\tilde{a}(x^4 - x^3)]| + \cosh[\tilde{a}(x^4 - x^3)] \right\} \right], \\ B &= \frac{1}{\tilde{a}} e^c \left[\frac{a}{\tilde{a}} \sqrt{\tilde{a}^2 + 1} |\sinh^2[\tilde{a}(x^4 - x^3)]| - \frac{b}{\tilde{a}} \right], \\ C &= -\frac{1}{\tilde{a}} e^{-c} \left[\frac{a}{\tilde{a}} + \sqrt{\tilde{a}^2 + 1} \right. \\ &\quad \times \left. \left\{ \frac{b}{\tilde{a}} |\sinh[\tilde{a}(x^4 - x^3)]| - \cosh[\tilde{a}(x^4 - x^3)] \right\} \right], \\ D &= e^{p[(d/p)x^4 + x^3]}. \end{aligned} \quad (20)$$

The solution is a gravitational pulse of an unchanged shape, moving at the speed of light.

This solution cannot be generated by the inverse scattering problem method (ISPM) formulated by Belinski and Zakharov [7].

5 Method of functional parameter. The first class of exact solutions

Other way of studying Eqs. (8) may be presented as follows. Solutions of (8) will be sought in the form

$$\theta = \theta(\xi), \quad \chi = \chi(\xi), \quad \phi = \phi(\xi), \quad \psi = \ln \xi \quad (21)$$

where $\xi = \xi(z, t)$ is a functional parameter satisfying

$$\square \xi = 0, \quad (22)$$

according to the first equation of (8).

Substitution of (21) to the other equation of (8), leads to the set of ordinary differential equations

$$\begin{aligned} \ddot{\theta} + \frac{1}{\xi} \dot{\theta} - \frac{1}{2} \dot{\chi}^2 \sinh 2\theta &= 0, \\ \ddot{\chi} + \frac{1}{\xi} \dot{\chi} + 2\dot{\chi} \dot{\theta} \coth \theta &= 0, \\ \ddot{\phi} + \frac{1}{2\xi^2} - \frac{1}{2} \dot{\theta}^2 - \frac{1}{4} \dot{\chi}^2 \sinh^2 \theta &= 0, \end{aligned} \quad (23)$$

for the functions θ , χ and ϕ of the parameter ξ treated as an independent variable. Here and henceforth

$$\dot{\phi}_a = \frac{d}{d\xi} \phi_a, \quad \ddot{\phi}_a = \frac{d^2}{d\xi^2} \phi_a, \dots; \quad a = 1, 2, 3, 4.$$

Eqs. (23) are exactly integrated and the solution may be written as

$$\begin{aligned} \theta_{\pm} &= \ln \left\{ \frac{a}{2} \left[\left(\frac{\xi}{\xi_0} \right)^k + \left(\frac{\xi}{\xi_0} \right)^{-k} \right] \right. \\ &\quad \left. \pm \sqrt{\frac{a^2}{4} \left[\left(\frac{\xi}{\xi_0} \right)^{2k} + \left(\frac{\xi}{\xi_0} \right)^{-2k} \right] + \frac{a^2}{2} - 1} \right\}, \\ \chi_{\pm} &= \chi_0 \pm \arctan \left\{ \frac{k}{c} \frac{(\xi/\xi_0)^k - (\xi/\xi_0)^{-k}}{(\xi/\xi_0)^k + (\xi/\xi_0)^{-k}} \right\}, \\ \phi_{\pm} &= \phi_0 + \phi_1 \xi - \left(\frac{k^2}{2} - 2 \right) (\ln \xi - 1) \\ &\quad - \frac{c}{4} \xi \int^{\xi} \frac{\chi_{\pm}(\xi')}{\xi'} d\xi', \\ a &= \sqrt{\frac{k^2 + c^2}{k^2}}, \end{aligned} \quad (24)$$

where ϕ_0 , ϕ_1 , χ_0 , ξ_0 , k , c are arbitrary real constants.

The above pairs of solutions $\{\theta_{\pm}(\xi), \chi_{\pm}(\xi), \phi_{\pm}(\xi), \psi(\xi)\}$ of (23) correspond to each solution $\xi = \xi(z, t) = f(z - t) + g(z + t)$ of Eq. (22) ($f(z - t)$ and $g(z + t)$ are arbitrary twice differentiable functions). The connection (7) between the metric coefficients and the chiral fields allows one to construct solutions of the Einstein equations (4) for the space-time (3).

The condition $\xi > 0$ is necessary for the field ϕ to be a real function.

The transformations (7), (8), (24) may be interpreted as a Bäcklund transformation of Eq. (22) to (4) for the space-time (3) since ξ is a functional parameter.

It is necessary to note that the Bäcklund transformation possesses an obvious functional form. Therefore in this case no additional construction is needed for obtaining the solutions, as it was, for example, in the case of the ISPM of Belinski-Zakharov [7]. The same shortcoming of the ISPM does not allow one to formulate in a simple way the conditions of coincidence between the Belinski-Zakharov family of solutions and those obtained here.

The possibility of formulating the initial and boundary-value problems for the gravitational field and constructing exact solutions is one more advantage of this method over the ISPM.

6 Method of functional parameter. Second class of exact solutions

Other family of solutions which cannot be reduced to the above family, may be obtained under the assumptions

$$\theta = \theta(\xi), \quad \chi = \chi(\xi), \quad \phi = \phi(\xi), \quad \psi = \text{const.} \quad (26)$$

Equations for $\theta(\xi)$, $\chi(\xi)$ and $\phi(\xi)$ are written in this case in the following way:

$$\begin{aligned} \ddot{\theta} - \frac{1}{2} \dot{\chi}^2 \sinh 2\theta &= 0, \\ \ddot{\chi} + 2\dot{\chi} \dot{\theta} \coth \theta &= 0, \\ \ddot{\phi} - \frac{1}{2} \dot{\theta}^2 - \frac{1}{4} \dot{\chi}^2 \sinh^2 \theta &= 0. \end{aligned} \quad (27)$$

Their solutions are constructed in a way similar to that described above. They are

$$\begin{aligned} \theta_{\pm} &= \ln \left\{ a \cosh k(\xi - \xi_0) \pm \sqrt{a^2 \cosh^2 k(\xi - \xi_0) - 1} \right\}, \\ \chi_{\pm} &= \int \frac{cd\xi}{\sinh^2 \theta_{\pm}} = \chi_0 \pm \arctan \left\{ \frac{k}{c} \tanh[k(\xi - \xi_0)] \right\}, \\ \phi_{\pm} &= \phi_0 + \phi_1 \xi + \frac{k^2}{2} \xi^2 - \frac{c}{4} e^{\xi} \int^{\xi} e^{-\xi'} \chi_{\pm}(\xi') d\xi'. \end{aligned} \quad (28)$$

The constant a is given by (25).

The difference between this family of solutions and the previous one is in the substitution of the parameter ξ to θ and χ instead of $\zeta = \ln \xi$ and in the addition to ϕ of a term depending on ξ^2 .

A comparison of the solutions (28) and (19) shows that the solution obtained using the isometric ansatz method does not belong to the family (28) for any value of the arbitrary constants and for any form of the function $\xi(x^3, x^4)$.

7 Generalization of the method of functional parameter

The idea of introducing a functional parameter may be generalized as follows. Consider one more family of substitutions, namely, when the chiral fields' dependence on z and t is determined by two functional parameters ξ and η , satisfying the equations

$$\xi_z = \eta_t, \quad \xi_t = \eta_z \quad (29)$$

and

$$\begin{aligned} \theta &= \theta(\xi, \eta), & \chi &= \chi(\xi, \eta), \\ \phi &= \phi(\xi, \eta), & \psi &= \text{const}. \end{aligned} \quad (30)$$

A solution to (8) will be found in the form (26). As follows from (29),

$$\square \xi = 0, \quad \square \eta = 0, \quad \xi_z \eta_z - \xi_t \eta_t = 0. \quad (31)$$

It leads to

$$\begin{aligned} \xi(z, t) &= g(z+t) + f(z-t), \\ \eta &= g(z+t) - f(z-t). \end{aligned} \quad (32)$$

It is possible to obtain a set of differential equations for $\theta(\xi, \eta)$, $\chi(\xi, \eta)$ and $\phi(\xi, \eta)$ by substituting (30) to Eqs. (8) with (31):

$$\begin{aligned} \ddot{\theta} - \frac{1}{2} \dot{\chi}^2 \sinh 2\theta &= 0, \\ \ddot{\chi} + 2\dot{\theta} \dot{\chi} \coth \theta &= 0, \\ \ddot{\phi} - \frac{1}{2} \dot{\theta}^2 - \frac{1}{4} \dot{\chi}^2 \sinh^2 \theta &= 0, \end{aligned} \quad (33)$$

$$\begin{aligned} \theta'' - \frac{1}{2} \chi'^2 \sinh 2\theta &= 0, \\ \chi'' + 2\chi' \theta' \coth \theta &= 0, \\ \phi'' - \frac{1}{2} \theta'^2 - \frac{1}{4} \chi'^2 \sinh^2 \theta &= 0. \end{aligned} \quad (34)$$

Here

$$\phi'_a = \frac{d}{d\eta} \phi_a, \quad \phi''_a = \frac{d^2}{d\eta^2} \phi_a, \dots, \quad a = 1, 2, 3, 4.$$

Eqs. (33) and (34) may be integrated independently over ξ and η and the resulting solutions have the same form as (28). But in this case the real functions ϕ_0 , ϕ_1 , ξ_0 , χ_0 , k , c depend on ξ for (34) and depend on η for (33).

The symmetry of the solutions with respect to the substitution of ξ to η makes it possible to write the

simplest form of the solutions, satisfying the continuity condition for the dependence of ϕ_a on the parameters ξ and η . To perform it in (28), it is necessary to use

$$\begin{aligned} k &= k_0(\eta - \eta_0), \quad c = c_0(\eta - \eta_0), \quad \phi_1 = p_0(\eta - \eta_0), \\ \phi_0, \chi_0, c_0, k_0, p_0, \eta_0, \xi_0 &= \text{const}. \end{aligned}$$

We should note that the solution obtained for a given nonlinear σ model is a trivial generalization of the one-parametric solution (28). The arbitrary functions $f(z-t)$ and $g(z+t)$ in (32) allow one to make the substitution

$$\begin{aligned} \xi &= f(z-t) + g(z+t) \rightarrow \xi = F(z-t) + G(z+t), \\ F(z-t) &= f^2(z-t) - (\xi_0 + \eta_0)f(z-t), \\ G(z+t) &= -g^2(z+t) + (\xi_0 - \eta_0)g(z+t), \end{aligned} \quad (35)$$

which leads to a coincidence between (32) and (28).

The problem of a two-parameter nontrivial solution for the above σ model is still open.

The problem may be solved by studying the equations for the functions ϕ_0 , ϕ_1 , ξ_0 , χ_0 , k , c depending on ξ and η , following from the coincidence of the solutions for the fields ϕ_0 , ϕ_1 , ξ_0 , χ_0 , k , c satisfying (33) and (34). This problem is perhaps connected with the chiral space structure and symmetries. It may play an important role in the analysis of the nonlinear σ model for other chiral space metrics.

8 Conclusions

The main result of the present paper may be formulated as follows.

The isometric ansatz method, derived from a symmetry investigation, allows one to obtain an exact solution for the chiral σ model of plane-symmetric gravitational vacuum spaces. Neither the inverse scattering problem method [7], nor the functional parameter method described here allow one to obtain such a solution. But the isometric ansatz cannot cover the whole family of possible solutions for the given σ model, as in the case of monopole family solutions [8].

A new family of solutions for the effective nonlinear chiral σ model of plane-symmetric gravitation vacuum spaces is obtained.

The explicit form of the solution yielded by the functional parameter method is obtained in a simpler way than using the Belinski-Zakharov method based on the ISPM. The solutions obtained may be used for formulating initial and boundary-value problems for the gravitational field of plane-symmetric vacuum spaces due to a very wide family of functions in the solution constructed by this method.

The functional parameter method is equivalent to the Bäcklund transformation of the d'Alembert equation to the gravitational field equations for plane-

symmetric space-times. This equivalence may be used for obtaining new exact solutions for nonlinear σ models corresponding to other classes of chiral space metrics.

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References

- [1] S.V. Chervon, “Non-Linear Fields in the Theory of Gravitation and Cosmology”, Middle-Volga Scientific Centre, Ulyanovsk State University, 1997.
- [2] R.A. Matzner and C.W. Misner, *Phys. Rev.* **154**, 1229 (1967).
- [3] S.V. Chervon and A.G. Muslimov, *Phys. Lett.* **A 142**, 14 (1989)
- [4] S.V. Chervon and A.G. Muslimov, “The Plane-Symmetric Gravitational Field as a Four-Component Non-Linear σ Model”, IPTI, Leningrad, No. 1347 (1989).
- [5] S.V. Chervon, *Grav. & Cosmol.* **3**, No. 2, 145 (1997).
- [6] G.G. Ivanov, *Teor. Mat. Fiz.* **57**, No. 1, 45 (1983) (in Russian).
- [7] A.V. Belinski and V.E. Zakharov, *Zh. Eksper. Teor. Fiz.* **75**, No. 6, 1953 (1978).
- [8] A.A. Belavin and A.M. Polyakov, *Pis'ma v Zh. Eksper. Teor. Fiz.* **22**, No. 10, 503 (1975).